AFCRL-62-48

NEW YORK UNIVERSITY

Courant Institute of Mathematical Sciences

Division of Electromagnetic Research

RESEARCH REPORT No. EM-173

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Contract No. AF 19(604)3495

MARCH, 1962

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Project 5631

Task 563104

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The research reported in this document was sponsored by the Electronics Research Directorate, Air Force Cambridge Research Laboratories, Office of Aerospace Research, USAF, Beford, Mass. 

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## ABSTRACT

In this paper an attempt has been made to clarify certain questions about the convergence of the Bremmer series solution[2] of the linear second order differential equation

$$u'' + k^2(x)u = 0 .$$

These questions have been brought forth recently in mathematical literature, in particular in an article by F. V. Atkinson[1] and in an earlier one by Bellman and Kalaba.[5] It is shown here that series of the Bremmer type, other than that discussed by Atkinson, exist and have better convergence properties. Moreover, such series have been given by J. B. Keller and H. B. Keller[8] for the more general case of a system of first order differential equations.

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## I. Introduction

F. V. Atkinson,[1] in an elegant analysis, has investigated the validity of a multiply reflected wave series solution of the differential equation

$$u'' + k^2(x)u = 0 . \tag{1.1}$$

The solution has the form

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots , \tag{1.2}$$

each term of which corresponds to a wave traveling to the left or to the right. The solution is assumed to be one such that the sum of all the left moving waves vanishes in the limit $x \to +\infty$. The terms are classified according to the number of reflections the corresponding wave has undergone.

The solution (1.2) was given independently by Bremmer,[2] Landauer,[3] and Schelkunoff.[4] It was discussed by Bellman and Kalaba[5] who also considered its validity. Atkinson[1] showed that if $k(x)$ is continuously differentiable and if $k(x)$ satisfies the conditions*

$$k(x) \geq a > 0 , \qquad \int_0^\infty |k'(x)|dx < \infty , \tag{1.3-4}$$

where $\underline{a}$ is a constant, as well as

$$\int_0^\infty |k'(x)/k(x)|dx \leq \pi , \tag{1.5}$$

the series (1.2) converges. He also showed that the number $\pi$ in (1.5) cannot

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*The second condition actually follows from the first and (1.5).

be replaced by any larger number.

The solution of (1.1) investigated by Atkinson is not the most efficient multiply reflected wave series solution from the point of view of convergence properties, however. A more rapidly converging series solution of the same type exists [6] and for this solution the condition (1.5) can be replaced by the weaker condition

$$\int_0^\infty |k'(x)/k(x)|dx < \infty . \tag{1.6}$$

Moreover, the construction of this more rapidly converging series can be generalized in a straightforward way to the case of a system of first order differential equations. An analogous series solution for the system can be constructed, and the condition for the convergence of this series is correspondingly as weak as (1.6) in an appropriate sense. This series solution for the system converges more rapidly and under weaker conditions than a similar one investigated by Atkinson [1] who also generalized (1.1) and (1.2) to the case of a system of ordinary differential equations. A general procedure for constructing a multiply reflected wave series solution for a system of differential equations is essentially contained in the work of Keller and Keller, [8] of which the series in [6] is a special case.

It is the purpose of the present remarks to call attention to the existence of these alternative multiply reflected wave series solutions so that any false impression concerning the limitations of this type of solution may be dispelled. Actually, a slight modification of the series will provide a solution which is valid even when k(x) has finite jump discontinuities. [7]

## II. Some Definitions and Relations

The following are a set of definitions and results used by Atkinson[1] and repeated here for convenience:

$$u(x) = v(x) + w(x) \ , \tag{2.1}$$

with $v(x)$ representing the total contribution of right-moving waves and $w(x)$ representing the total contribution of left-moving waves; $y(x)$ is a solution of the differential equation

$$y' = (ik - \tfrac{1}{2}k^{-1} k')y \ , \tag{2.2}$$

$$y(x) = [k(x)]^{-1/2} \exp\left[i \int_0^x k(x)ds\right] \ ; \tag{2.3}$$

$z(x)$ is a solution of the differential equation

$$z' = (-ik - \tfrac{1}{2}k^{-1} k')z \ , \tag{2.4}$$

$$z(x) = [k(x)]^{-1/2} \exp\left[-i \int_0^x k(s)ds\right] \ ; \tag{2.5}$$

there are also defined the functions

$$y(t_2,t_1) = y(t_2)y^{-1}(t_1) \ , \tag{2.6}$$

$$z(t_2,t_1) = z(t_2)z^{-1}(t_1) \ . \tag{2.7}$$

The functions $v(x)$ and $w(x)$ satisfy the integral equations

$$v(x) = y(x) + \frac{1}{2}\int_0^x y(x,s)k^{-1}(s)k'(s)w(s)\,ds\ , \tag{2.8}$$

$$w(x) = -\frac{1}{2}\int_x^\infty z(x,s)k^{-1}(s)k'(s)v(s)\,ds\ . \tag{2.9}$$

They also satisfy the differential equations

$$v' = (ik - \frac{1}{2}k^{-1}k')v + \frac{1}{2}k^{-1}k'w\ , \tag{2.10}$$

$$w' = (-ik - \frac{1}{2}k^{-1}k')w + \frac{1}{2}k^{-1}k'v\ , \tag{2.11}$$

which together with (2.1) imply (1.1).

## III. The Multiply Reflected Wave Series and Convergence

The series whose validity was investigated by Atkinson[1] is the Neumann series solution of (2.8-9). This solution can also be obtained by introducing a parameter $\lambda$ in front of each integral in (2.8-9), assuming series expansions for $v$ and $w$ in powers of $\lambda$ and then setting $\lambda$ equal to one. The series solutions are

$$v(x) = \sum_{n=0}^{\infty} u_{2n}(x)\ , \tag{3.1}$$

$$w(x) = \sum_{n=0}^{\infty} u_{2n+1}(x)\ , \tag{3.2}$$

where the terms of (3.1-2) satisfy the recursion relations

$$u_{2n+1}(x) = -\frac{1}{2}\int_x^\infty z(x,s)k^{-1}(s)k'(s)u_{2n}(s)\,ds\ , \qquad (3.3)$$

$$u_{2n}(x) = \frac{1}{2}\int_x^\infty y(x,s)k^{-1}(s)k'(s)u_{2n-1}(s)\,ds \qquad (3.4)$$

and $u_o(x) = y(x)$. The conditions (1.3-5) on the convergence of (3.1-2) were given by Atkinson.[1]

In Section IV of this paper a general method for obtaining a multiply reflected wave series solution of a system of first order differential equations will be described. This method leads in a natural way to integral equations from which the series can be obtained. Thus, if the second order differential equation (1.1) is replaced by an equivalent pair of first order differential equations

$$u' = -if\ ,$$

$$f' = -ik^2u\ ,$$

where the functions $u(x)$ and $f(x)$ are defined in terms of $v(x)$ and $w(x)$ by

$$u = w + v\ ,$$

$$f = k(w - v)\ ,$$

then the preliminary steps of the general method described in Section IV of this paper will lead directly to (2.10-11). The final steps of that method require the definitions

$$\tilde{v}(x) = y(x)\,\tilde{p}(x) ,$$

$$\tilde{w}(x) = z(x)\,\tilde{q}(x) ,$$

where $\tilde{v}(x)$ and $\tilde{w}(x)$ are a solution pair of (2.10-11). When these are substituted into (2.10-11) and use is made of (2.2-4) there results the system of differential equations

$$\tilde{p}' = \tfrac{1}{2}k^{-1}k'y^{-1}z\tilde{q} ,$$

$$\tilde{q}' = \tfrac{1}{2}k^{-1}k'z^{-1}y\tilde{p} .$$

By integrating both sides of these equations one can obtain, in particular,

$$\tilde{p}(x) = c - \frac{1}{2}\int_x^\infty k^{-1}k'y^{-1}z\tilde{q}\,ds ,$$

$$\tilde{q}(x) = -\frac{1}{2}\int_x^\infty k^{-1}k'z^{-1}y\tilde{p}\,ds ,$$

where c is an arbitrary constant. This leads at once to a pair of integral equations for $\tilde{v}(x)$ and $\tilde{w}(x)$:

$$\tilde{v}(x) = cy(x) - \frac{1}{2}\int_x^\infty y(x,s)k^{-1}(s)k'(s)\tilde{w}(s)\,ds , \qquad (3.5)$$

$$\tilde{w}(x) = -\frac{1}{2}\int_x^\infty z(x,s)k^{-1}(s)k'(s)\tilde{v}(s)\,ds . \qquad (3.6)$$

Observe that $\tilde{w}(x)$ approaches zero as x approaches infinity; this same condition is satisfied by w(x) in (2.8-9). If the constant c is selected appropriately the solutions $\tilde{v}(x)$, $\tilde{w}(x)$ of (3.5-6) can be made identical with the solutions v(x), w(x) of (2.8-9). From a consideration of $\lim_{x \to 0} [\tilde{v}(x)/y(x)] = \lim_{x \to 0} [v(x)/y(x)] = 1$ it can be seen that the proper choice is

$$c = 1 + \frac{1}{2}\int_0^\infty y^{-1}(s)k^{-1}(s)k'(s)\tilde{w}(s)\, ds . \tag{3.7}$$

In order to obtain a series solution of (3.5-6) one can use a Neumann expansion starting with $\tilde{v}_o(x) = cy(x)$. The recursion formulas for the terms of the series are

$$\tilde{u}_{2n}(x) = -\frac{1}{2}\int_x^\infty y(x,s)k^{-1}(s)k'(s)\tilde{u}_{2n-1}(s)\, ds , \tag{3.8}$$

$$\tilde{u}_{2n+1}(x) = -\frac{1}{2}\int_x^\infty z(x,s)k^{-1}(s)k'(s)\tilde{u}_{2n}(s)\, ds . \tag{3.9}$$

The series solutions of (3.5-6) are then

$$\tilde{v}(x) = \sum_{n=0}^{\infty} \tilde{u}_{2n}(x) , \tag{3.10}$$

$$\tilde{w}(x) = \sum_{n=0}^{\infty} \tilde{u}_{2n+1}(x) . \tag{3.11}$$

It can be shown by means of a standard procedure [6] that the Neumann series for (3.5-6) converges rapidly under the condition (1.6), for the integral

equations (3.5-6) are of the Volterra type. It is possible also to prove this convergence property by means of a majorizing series of the type used by Atkinson. [1]

Following Atkinson one can define

$$\tilde{p}(x) = y^{-1}(x)\tilde{v}(x), \qquad \tilde{q}(x) = z^{-1}(x)\tilde{w}(x)$$

so that the integral equations (3.5-6) become

$$\tilde{p}(x) = c - \frac{1}{2}\int_x^\infty y^{-1}k^{-1}k'z\tilde{q}\,ds \tag{3.12}$$

$$\tilde{q}(x) = -\frac{1}{2}\int_x^\infty z^{-1}k^{-1}k'y\tilde{p}\,ds\ , \tag{3.13}$$

and the recursion relations (3.8-9) become

$$\rho_{2n}(x) = -\frac{1}{2}\int_x^\infty y^{-1}k^{-1}k'z\rho_{2n-1}ds\ , \tag{3.14}$$

$$\rho_{2n+1}(x) = -\frac{1}{2}\int_x^\infty z^{-1}k^{-1}k'y\rho_{2n}ds\ . \tag{3.15}$$

The solutions $\rho_n(x)$ of (3.14-15) are terms in the series

$$\tilde{p}(x) = \sum_{n=0}^{\infty}\rho_{2n}(x)\ , \tag{3.16}$$

$$\tilde{q}(x) = \sum_{n=0}^{\infty}\rho_{2n+1}(x)\ . \tag{3.17}$$

The convergence of the series (3.16-17) depends upon the convergence of

$$\sum_{n=0}^{\infty} \rho_n(x)\lambda^n$$

in the unit circle in the complex $\lambda$ plane. As in Atkinson's analysis this series is majorized by

$$\sigma(x) = \sum_{n=0}^{\infty} \sigma_n(x)\lambda^n , \tag{3.18}$$

where $\sigma_o(x) = c$ and the remaining $\sigma_n(x)$ are defined by

$$\sigma_{2n}(x) = \frac{1}{2} \int_x^{\infty} |k^{-1}k'| \, \sigma_{2n-1} ds , \tag{3.19}$$

$$\sigma_{2n+1}(x) = \frac{1}{2} \int_x^{\infty} |k^{-1}k'| \, \sigma_{2n} ds . \tag{3.20}$$

The absolute convergence of (3.18) is equivalent to that of

$$\hat{\sigma}(x) = \sum_{n=0}^{\infty} \sigma_{2n}(x)\lambda^{2n} . \tag{3.21}$$

The series (3.21) can be summed through a consideration of the differential equation which it satisfies and the conditions

$$\hat{\sigma}(\infty) = c , \qquad \lim_{x \to \infty} \left\{ \frac{\hat{\sigma}'}{|k^{-1}k'|} \right\} = 0 .$$

The sum is found to be

$$\hat{\sigma}(x) = c \cosh\left(\frac{1}{2}\lambda \int_x^\infty |k^{-1}k'|ds\right) . \qquad (3.22)$$

Since this function is an entire function of $\lambda$ the majorizing series converges for all $\lambda$ and in particular for $\lambda = 1$. Thus, the series (3.16-17) converges under the condition (1.6).

## IV. The Case of a System of Differential Equations*

The differential equations (2.10-11) for a propagating wave can be generalized to a system of differential equations which can then be replaced by a system of Volterra integral equations. Let $A(x)$ be an n by n matrix and $u(x)$ an n dimensional column vector. Consider the differential equation

$$u'(x) = A(x)u(x) . \qquad (4.1)$$

Let $K(x)$ be an n by n matrix which is nonsingular for almost all values of x such that $K^{-1}AK$ is diagonal with diagonal elements $\kappa_n(x)$. After the substitution

$$u(x) = K(x)v(x)$$

there results in place of (4.1) the differential equation

$$v'(x) = \left[-K^{-1}(x)K'(x) + K^{-1}(x)A(x)K(x)\right]v(x) . \qquad (4.2)$$

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*The solution given in this section is due to Keller and Keller.[8]

The differential equation (4.2) is a generalization of the system (2.10-11).

Let the diagonal elements of the matrix $K^{-1}(x)K'(x)$ be $\gamma_n(x)$. Then define the matrix $Y(x)$ as the solution of the differential equation

$$Y'(x) = \left[-\Gamma(x) + K^{-1}(x)A(x)K(x)\right]Y(x) , \qquad (4.3)$$

where $\Gamma(x)$ is the diagonal part of $K^{-1}(x)K'(x)$. The matrix $Y(x)$ is diagonal and can be calculated explicitly since the matrix factor of $Y(x)$ on the right of (4.3) is diagonal. The solution $Y(x)$ is determined except for an arbitrary constant diagonal matrix factor which can be taken to be the identity. The n-th diagonal element of $Y(x)$ will have the form

$$Y_n(x) = \exp\left[-\int_0^x \gamma_n(s)ds + \int_0^x \kappa_n(s)\, ds\right] . \qquad (4.4)$$

Now define the vector $p(x)$ by

$$v(x) = Y(x)p(x)$$

and substitute into (4.2). There results, with the use of (4.3), the differential equation

$$p'(x) = Y^{-1}(x)H(x)Y(x)p(x) \qquad (4.5)$$

where $H(x)$ is $K^{-1}(x)K'(x)$ with each of its diagonal elements replaced by zero. An integral equation for $p(x)$ can be obtained from (4.5) by integrating both sides

$$p(x) = c - \int_x^\infty Y^{-1}(s)H(s)Y(s)ds , \qquad (4.6)$$

where c is an arbitrary constant vector.

Left-moving and right-moving wave components of the wave vector v(x) can be defined according to whether the imaginary part of the corresponding quantity $\int_0^x \kappa_n(x)ds$ in (4.4) is a decreasing or an increasing function of x. To ensure that no left-moving wave components are present at $x = \infty$ one can choose the constant vector c in (4.6) so that its components corresponding to the left-moving wave components of v(x) are zero.

The integral equation (4.6) is of the Volterra type; therefore, its Neumann series expansion converges rapidly and under relatively weak conditions. If the elements of a matrix M are $M_{ij}$, a convenient norm for the matrix is

$$||M|| = \left[\sum_j \left(\sum_i |M_{ij}|\right)^2\right]^{1/2} . \qquad (4.7)$$

For the series solution of (4.6) the condition for convergence which corresponds to (1.6) is [6]:

$$\int_0^\infty ||Y^{-1}(s)H(s)Y(s)||\,ds < \infty . \qquad (4.8)$$

When the differential equation (4.1) describes an energy conserving propagating wave the diagonal matrix Y(s) is unitary, and in this case the condition (4.8) becomes

$$\int_0^\infty ||H(x)||\,ds < \infty . \qquad (4.9)$$

It is interesting to note that if the frequency $\omega$ is displayed by replacing the matrix $A(x)$ in (4.1) with $\omega A(x)$, the error estimate of any partial sum of the series solution of (4.1) based on the Neumann series solution of (4.6) depends only on the quantity

$$\int_0^\infty \| H(s) \| \, ds$$

and thus is independent of $\omega$. The implication of this fact is that the series solution will converge uniformly with respect to $\omega$.

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